University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 3

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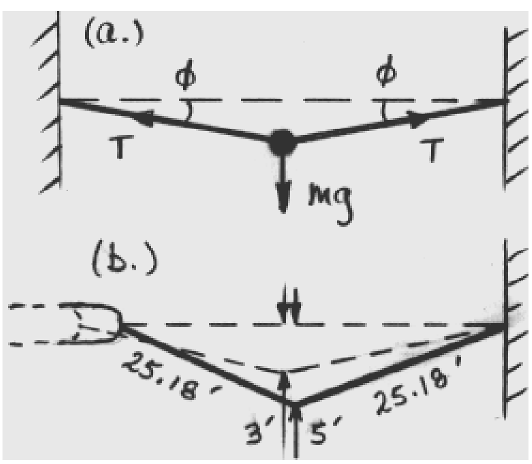
1. (a.) This problem is a simple force balance. The component of the tension in the rope pointing up must balance the force of gravity pulling down on the shirt. We are going to ignore the effect of gravity on the rope. The angle that the rope makes with the horizontal is just $\tan\phi = 8$ cm/5 m $= 0.016$. The sine of this angle is very close to this value. In fact $\sin\phi = 0.015998$, so we will use $\sin\phi = 0.016$.

The equation for the force balance in the vertical direction is just

$$2T\sin\phi = mg$$

This ensures that the force of tension balances gravity. There is a factor of two in front of the tension because the tension in each half of the rope acts on the mass. Solving this equation with the values given, the answer is

$$T = 153 \text{ N}$$



(b.) This part is similar to the first. In this case the force being applied to the rope is a man pushing on it, no gravity, but the method is the same. The force of 500 N must be balanced by tension in the rope. The angle $\tan\phi = 3/25 = 0.12$ is much larger, but the approximation $\sin\phi \approx \tan\phi \approx \phi$ is still very accurate, so we will use $\sin\phi = 0.12$. We also need $\cos\phi = 0.99$. The force of tension is found by a force balance

$$2T\sin\phi = 500 \text{ N} \;\Rightarrow\; T = 2083 \text{ N}$$

The force acting on the car is just the cosine of the angle times this tension, assuming that we want the force directed towards the tree.

$$F = T\cos\phi = 2062 \text{ N}$$

Now we want to find how far the car is shifted. The original right triangle had sides 3 ft, 25 ft, and $\sqrt{9+625}$ =25.18 ft. The rope will not stretch anymore, so the hypotenuse of this triangle will remain the same, 25.18 ft. The rope is pushed an additional 2 ft, so the short side now has length 5 ft. The long side must have length $\sqrt{634-25} = 24.68$ ft. The car has moved twice this distance, since the are two triangles with total hypotenuse ≈ 50 ft; only 0.63 ft of car movement is produced by pushing the rope 2 ft. This isn't a very practical way to get the car out of the ditch.

2. (a.) In this problem we will again use the fact that $\sin\theta \approx \theta$ when $\theta \ll 1$. The angle that the rope makes with the normal force is $\Delta\theta/2$, so the tension in the normal direction is $T\sin(\Delta\theta/2) \approx T\Delta\theta/2$. There are two tensions here, one from each side of the rope, and they must balance the normal force. The normal force on this section of string is then just

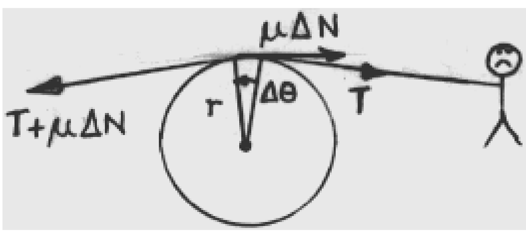
$$\Delta N = T\Delta\theta$$

(b.) The length of rope that covers an angle $\Delta\theta$ on a circular object is just $r\Delta\theta$. θ of course is measured in radians. The normal force per length on the cylinder is $\Delta N/\Delta\ell$. Plugging in the result for ΔN from part (a.), and using $\Delta\ell = r\Delta\theta$, we find

$$\frac{\Delta N}{\Delta\ell} = \frac{T}{r}$$

(c.) When the tension on the rope is not constant, there can be slipping. The force countering this is friction. If the tension changes by an amount ΔT along a small section of rope, the frictional force must be equal to it. The frictional force is $\mu \Delta N$ when it is just about to slip. We thus get $\Delta T = \mu \Delta N$. Plugging in the results of (a.), we get $\Delta T = \mu T \Delta\theta$. We are going to promote this relation between small quantities to a differential relation, so that we can get a differential equation to solve.

$$dT = \mu T d\theta \;\Rightarrow\; \frac{dT}{d\theta} = \mu T$$



This is a differential equation that we can solve by direct integration.

$$\frac{dT}{d\theta} = \mu T \;\Rightarrow\; \frac{dT}{T} = \mu d\theta$$
$$\Rightarrow \int_{T_0}^{T(\theta)} \frac{dT'}{T'} = \int_0^{\theta} \mu d\theta'$$

These integrals are ones that you should memorize if you haven't yet. The result is

$$\ln T(\theta) - \ln T_0 = \mu\theta \;\Rightarrow\; T(\theta) = T_0 e^{\mu\theta}$$

The tension increases exponentially, provided that the rope is about to slip.

(d.) Here we are going to calculate some values for this amplification of force. $\mu = 0.2$ and $T_0 = 100$ lbs. The values of T for 1, 2, 3, and 4 complete turns are as follows. One complete turn has angle 2π. $\text{Exp}(2\pi\mu) = 3.51$, so the tension at the other end is 351 lbs. For two turns, the angle is 4π, so the amplification factor is $\exp(4\pi\mu) = 12.35$ and the tension at the other end is 1235 lbs. For three complete turns, the tension is 4338 lbs. For four complete turns, the tension is 15240 lbs, almost 8 tons!

3. A dish sits in the middle of a square table of side s. The coefficient of friction between dish and tablecloth is μ_1. The coefficient of friction between the dish and table is μ_2. The tablecloth is rapidly pulled out from under the dish. The dish moves a distance x_1 while in contact with the moving tablecloth, and a distance x_2 while in contact with the table. Let the mass of the dish be m, but we will see that this doesn't matter.

(a.) The tablecloth is being pulled out from under the dish, so the dish is sliding on the tablecloth. The frictional force tends to pull the dish to the edge of the table because this opposes the direction of the sliding. The normal force of the dish on the table is just mg, so the force of sliding friction is just $\mu_1 mg$. Newton's second law then tells us that

$$F = ma = m\mu_1 g \;\Rightarrow\; a = \mu_1 g$$

This is a constant acceleration, so we can easily determine the amount of time that the dish is on the tablecloth and its maximum velocity. We know the total distance traveled is x_1, so we get the following equations for the time of contact t_1 and the maximum velocity v:

$$x_1 = \frac{1}{2}\mu_1 g t_1^2 \qquad v = \mu_1 g t_1$$

These equations are easily solved for t and v:

$$t_1 = \sqrt{\frac{2x_1}{\mu_1 g}} \qquad v = \sqrt{2x_1\mu_1 g}$$

(b.) We do the same thing for the period when the dish slides on the table. This time, the frictional force tends to slow the dish down. The frictional force is $-\mu_2 mg$, so the acceleration is $a = -\mu_2 g$. The trip starts at the maximum velocity v, and ends with the dish at rest having moved a distance x_2. We can again solve for the travel time t_2 and the maximum velocity v:

$$x_2 = -\frac{1}{2}\mu_2 g t_2^2 + v t_2 \qquad v = \mu_2 g t_2$$

Again these equations are easily solved

$$t_2 = \frac{v}{\mu_2 g} \qquad v = \sqrt{2x_2\mu_2 g} \qquad t_2 = \sqrt{\frac{2x_2}{\mu_2 g}}$$

(c.) In part (c.), we derived two expressions for the maximum velocity v. If we equate these we can get an expression relating the distances traveled x_1 and x_2.

$$\sqrt{2x_1\mu_1 g} = \sqrt{2x_2\mu_2 g} \;\Rightarrow\; \mu_1 x_1 = \mu_2 x_2$$

For the dish to remain on the table, we need the total distance traveled to be less than half the length of the table, $x_1 + x_2 \leq s/2$. We can combine these equations to solve for the distance x_1 that the dish spends on the tablecloth. We will consider the case where the dish stops at the edge of the table so the total distance traveled is $s/2$. From the previous equation we get a relation between x_1 and x_2

$$x_2 = \frac{\mu_1}{\mu_2} x_1$$

Combining this with the previous equation, we get the final answer

$$x_1 + x_2 = \frac{s}{2} \;\Rightarrow\; \left(1 + \frac{\mu_1}{\mu_2}\right) x_1 = \frac{s}{2}$$
$$\Rightarrow\; x_1 = \left(\frac{s}{2}\right)\frac{\mu_2}{\mu_1 + \mu_2}$$

(d.) To find the amount of time the dish is in contact with the tablecloth, we just use the result of part (a.), $t_1 = \sqrt{2x_1/\mu_1 g}$. Plugging in our result, we get

$$t_1 = \sqrt{\left(\frac{s}{g}\right)\left(\frac{\mu_2}{\mu_1}\right)\frac{1}{\mu_1 + \mu_2}}$$

(e.) It is possible that the dish won't slide at all. This is the case of static friction. This force must be overcome but a sufficiently hard tug. The tablecloth must accelerate at the beginning of the pull to get to its high, but constant velocity. If this acceleration is too low ($a_1 < \mu_1^{\text{static}} g$), the force of static friction will be sufficient to accelerate the dish at the same rate, and the host will not be pleased. If the initial acceleration is high enough, the force of static friction cannot impart the necessary acceleration to the dish, and it begins to slide.

4. A string has length L and can support a tension T. A mass m is spun around on the end of the string.

(a.) The string is spun horizontally. The centripetal acceleration is v^2/L, so the force needed to supply this acceleration is mv^2/L. In this case, the only force to consider is the tension. The maximum velocity is given by

$$\frac{mv^2}{L} = T \;\Rightarrow\; v = \sqrt{\frac{LT}{m}}$$

The above would be correct if gravity could be ignored for part (a.). However, we know this cannot be the case, because then there would be no definition for the word "horizontal". Gravity must be present. If so, the string makes an angle θ with the horizontal, and the radius of the spin is $L\cos\theta$. The centripetal acceleration is $v^2/L\cos\theta$. This must be provided entirely by the tension in the rope in the radial direction, which is $T\cos\theta$. This gives a relationship between v and θ for a given T:

$$v^2 = \frac{TL}{m}\cos^2\theta \;\Rightarrow\; \cos^2\theta = \frac{mv^2}{TL}$$

The tension must also oppose the force of gravity downward, giving a second relation, which will be easier to handle when squared

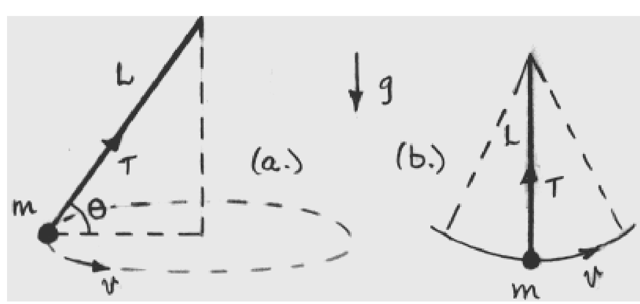
$$T\sin\theta = mg \;\Rightarrow\; \sin^2\theta = \left(\frac{mg}{T}\right)^2$$

Adding these two equations, and using $\cos^2\theta + \sin^2\theta = 1$, we get a condition on the velocity

$$\frac{m}{TL}v^2 + \frac{m^2 g^2}{L^2} = 1 \;\Rightarrow\; v^2 = \frac{LT}{m} - \frac{mg^2 L}{T}$$

The maximum velocity considering gravity is thus

$$v = \sqrt{\frac{LT}{m} - \frac{mg^2 L}{T}}$$



(b.) The string is spun vertically. In this case we must also consider gravity. Gravity directly opposes the tension when the mass is at its low point. At any other point, the tension is less. At the low point, the two forces of tension and gravity oppose each other, so the tension must be higher to provide the centripetal acceleration. When the rope is at an angle θ to the vertical, the centripetal force is provided by two sources, gravity and tension

$$\frac{mv^2}{L} = T(\theta) - mg\cos\theta$$

$T(\theta)$ is largest at the bottom of the path, when $\theta = 0$. Set $T(0) = T$, the largest allowed tension. This gives the final answer

$$\frac{mv^2}{L} = T - mg \;\Rightarrow\; v = \sqrt{\frac{LT}{m} - Lg}$$

5. K&K problem 2.31

Find the effective spring constants of these two spring systems. Remember that the frequency of oscillation is $\omega = \sqrt{k/m}$.

(a.) Consider the point where the springs are attached to each other. There shouldn't be any force acting there because the small point is massless. We can write equations for the displacements of the two springs x_1 and x_2. This spring force acts like a tension in that it pulls from both ends.

$$F_1 = -k_1x_1 = F_2 = -k_2x_2 \;\Rightarrow\; k_1x_1 = k_2x_2$$

The total displacement of the mass is $x_1 + x_2$, and the total force is just the spring force of the bottom spring $F = -k_2x_2$. Applying the general relation $F = -kx$, where k is the spring constant of the combined spring system and $x = x_1 + x_2$, we get

$$F = -k_2x_2 = -k(x_1 + x_2) \;\Rightarrow\; k = \frac{k_2x_2}{x_1 + x_2}$$

Using the fact that x_1 can be written in terms of x_2, we can remove the position dependence from the equation:

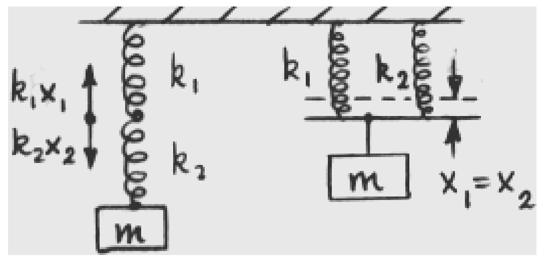
$$k = \frac{k_2}{\frac{k_2}{k_1} + 1} \;\Rightarrow\; k = \frac{k_1k_2}{k_1 + k_2}$$

This relationship is usually written in a different way that is easier to remember. You can check that it is correct:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

The final result for the frequency is

$$\omega_a = \sqrt{\frac{k_1k_2}{m(k_1 + k_2)}}$$



(b.) In this case the thing to notice is that the displacements of the two springs must be equal, otherwise the support would be tilting.

$$F_1 = -k_1x \qquad\qquad F_2 = -k_2x$$

The total force acting on the mass is just the sum of these two forces $F = F_1 + F_2$. We can easily find the effective spring constant of the system:

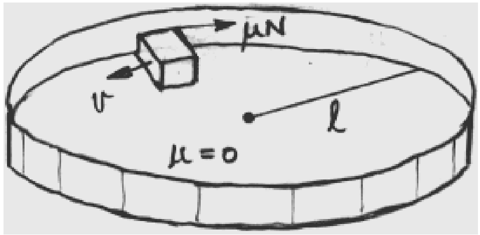
$$F = -kx = -k_1x - k_2x \;\Rightarrow\; k = k_1 + k_2$$

The final result for the frequency is

$$\omega_b = \sqrt{\frac{k_1 + k_2}{m}}$$

6. K&K problem 2.35

(a.) In this problem we need to solve a differential equation. A block slides on a frictionless table inside a fixed ring of radius l. The ring has a coefficient of friction μ. We want to find the velocity as a function of time. At time $t = 0$ the velocity is v_0. We will assume that the block moves in the circular path defined by the ring. This makes it effectively a one dimensional problem.



There are two forces acting on the block in the plane of the table. They are the normal force exerted by the ring and the frictional force. The normal force merely makes the block move in the circular path that we assumed. We do need to know it though, because we want to calculate the frictional force. We find it in the usual way, by requiring that it provide the centripetal acceleration.

$$a_{\text{centripetal}} = \frac{v^2}{l} \;\Rightarrow\; N = \frac{mv^2}{l}$$

We can now write the equation of motion for the particle by Newton's second law

$$m\frac{dv}{dt} = -\mu N = -\frac{\mu m v^2}{l} \;\Rightarrow\; \frac{dv}{dt} = -\frac{\mu}{l}v^2$$

This equation can be solved by direct integration, as you saw in problem 2:

$$\frac{dv}{v^2} = -\frac{\mu}{l} \;\Rightarrow\; \int_{v_0}^{v(t)} \frac{dv}{v^2} = -\int_0^t \frac{\mu\, dt}{l}$$
$$\Rightarrow\; \frac{1}{v_0} - \frac{1}{v(t)} = -\frac{\mu t}{l}$$

This result can be simplified to get K&K's result

$$v(t) = v_0\left(1 + \frac{\mu v_0 t}{l}\right)^{-1}$$

(b.) Now that we know the velocity of the block, finding the position is easy. It is easiest to describe the position in terms of the angle on the circle. We can easily determine the angular velocity as a function of time, because $\omega(t) = v(t)/l$. We know the velocity, so to get the position we just integrate. Assume that $\theta = 0$ at $t = 0$, which also means $x = l,\ y = 0$

$$\theta(t) = \int_0^t \omega(t')dt' = \int_0^t \frac{v_0}{l}\Big(\frac{1}{1 + \mu v_0 t'/l}\Big)dt'$$
$$= \frac{1}{\mu}\ln\Big(1 + \frac{\mu v_0}{l}t'\Big)_0^t$$

The final result for the total angle traveled is

$$\theta(t) = \frac{1}{\mu}\ln\Big(1 + \frac{\mu v_0}{l}t\Big)$$

Notice that the total distance traveled is infinite if one waits for an infinite time, even though the velocity approaches zero as time increases. We can of course write the x and y coordinates of the block as functions of time:

$$x(t) = l\cos(\theta(t)) \qquad\qquad y(t) = l\sin(\theta(t))$$

7. The two skaters each have mass 70 kg. Skater A carries a 10 kg bowling ball. Initially each skater is moving at 1 m/sec, and they are approaching each other. They are going to try to avoid collision by throwing the bowling ball back and forth.

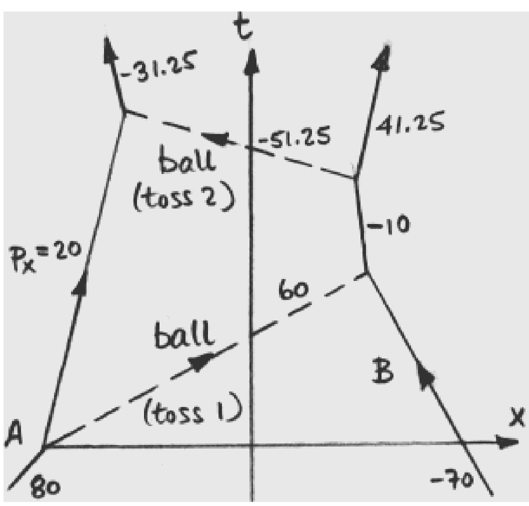
This problem uses conservation of momentum. Skater A starts out with $p_A = (70 + 10) \times 1 = 80$ kg-m/sec of momentum. Notice that we must include the momentum of the bowling ball in the momentum of skater A when he is carrying it. This adds 10 kg-m/sec to skater A's 70 kg-m/sec, since the bowling ball has a mass of $m = 10$ kg. Skater B is going in the opposite direction, so her momentum is negative, $p_B = -70$ kg-m/sec. Skater A throws the bowling ball to skater B in an attempt to stop the collision. Since there are no external forces on the system consisting of skater A and the bowling ball, the total momentum of these two objects is conserved. Throwing the bowling ball at 5 m/sec relative to the (initial) velocity of skater A gives it a momentum of 60 kg-m/sec. This is because the bowling ball velocity is 6 m/sec when we add the initial velocity of 1 m/sec. Skater A is left with a momentum of $p_A = 20$ kg-m/sec, so he hasn't reversed direction or stopped.

Next we consider the system of skater B and the bowling ball. Again there are no external forces, so the momentum of skater B plus the bowling ball is conserved. The total momentum is 60 kg-m/sec from the ball and -70

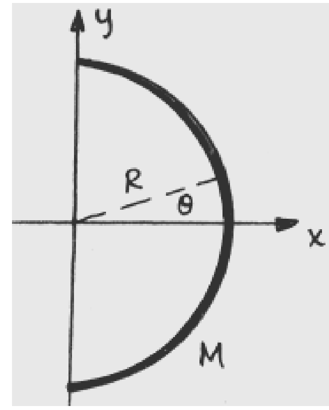
kg-m/sec from skater B. When skater B catches the ball, she will then have all of this momentum, $p_B = -10$ kg-m/sec. After this exchange, skater B has the ball, and the two skaters are still approaching each other. One toss was not enough. To summarize the first toss

$$\begin{aligned} \text{initial } p_A &= 80 \;\; p_B = -70 \\ \text{intermediate } p_A &= 20 \;\; p_B = -70 \;\; p_{\text{ball}} = 60 \\ \text{final } p_A &= 20 \;\; p_B = -10 \end{aligned}$$

The second toss will be enough to stop the collision. We calculate the velocity of skater B (including the bowling ball): $v = p/m = -0.125$ m/sec. Now skater B throws the bowling ball to skater A. The bowling ball's velocity will be -5.125 m/sec, so its momentum will be -51.25 kg-m/sec. This leaves skater B with $p_B = 41.25$ kg-m/sec. This is in the opposite direction to her initial motion. Skater A gets all of the momentum of the ball again, so $p_A = -31.25$ kg-m/sec. This is also opposite to his initial direction. So, after two tosses, the skaters are moving away from each other and the collision is averted. Plotting position versus time for the two skaters, we get a graph like the following:



8. This is an example of a center of mass calculation. Let's put the hoop on a polar coordinate system so that it goes from $\theta = (-\pi/2, \pi/2)$. In cartesian coordinates this is on the right half-plane.



It is fairly obvious that the center of mass lies on the line $Y_{\text{CM}} = 0$, or $\theta = 0$. The center of mass is calculated by the following

$$X_{\text{CM}} = \frac{1}{M} \int x \lambda \, dl$$

where λ is the linear mass density and dl is a differential of length on the hoop. If the hoop has mass M, the linear mass density is $\lambda = M/\pi R$. We can use polar coordinates to integrate this. Remember that $x = R\cos\theta$ for points on the hoop. The differential of length on the hoop is $dl = R d\theta$, so the integral we need to do is

$$X_{\text{CM}} = \frac{1}{M} \int_{-\pi/2}^{\pi/2} \frac{RM}{\pi} \cos\theta \, d\theta = \frac{2}{\pi} R$$